First-Order Methods for Nonsmooth Nonconvex Functional Constrained Optimization with or without Slater Points

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Problem of Interest

Consider

$$\begin{cases} \min_{x \in X} & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \qquad i = 1, \ldots, m. \end{cases}$$

- x: decision variable
- $X \subseteq \mathbb{R}^n$: convex and closed set (not necessarily bounded)
- $f(x), g_i(x)$: continuous and weakly convex (nonsmooth)

Weak Convexity of f and g_i

μ-Strongly Convex

$\forall x, x' \in X$ and $\forall \zeta \in \partial h(x)$, $\exists \mu > 0$, $h - \frac{\mu}{2}\|\cdot\|^2$ is convex, or equivalently

$$h(x') \geq h(x) + \zeta^T(x' - x) + \frac{\mu}{2}\|x' - x\|^2.$$

ρ-Weakly Convex

$\forall x, x' \in X$ and $\forall \zeta \in \partial h(x)$, $\exists \rho > 0$, $h + \frac{\rho}{2}\|\cdot\|^2$ is convex, or equivalently

$$h(x') \geq h(x) + \zeta^T(x' - x) - \frac{\rho}{2}\|x' - x\|^2.$$

Some Practical Scenarios

Nonsmooth and Weakly Convex Examples:

- Objective function
  - Phase retrieval: $f(x) = \frac{1}{m}\sum_{i=1}^{m} |(a_i^T x)^2 - b_i^2|$
  - Blind deconvolution: $f(x, y) = \frac{1}{m}\sum_{i=1}^{m} |(u_i^T x)(v_i^T y) - b_i|$
  - Robust PCA: $f(X, Y) = \|XY^T - M\|_1$
- Constraint function
  - Smoothly Clipped Absolute Deviation (SCAD) regularizer
- Classification Problems
  - Multi-class Neyman-Pearson classification
  - Classification with fairness constraints

Inexact Proximal Point Methods

For an unconstrained problem

$$x_{k+1} \approx \operatorname{argmin}_{x \in X} \left\{ f(x) + \frac{1}{2\alpha} \|x - x_k\|^2 \right\}$$

Extension to a constrained problem

$$x_{k+1} \approx \operatorname{argmin}_{x \in X} \left\{ f(x) + \frac{1}{2\alpha} \|x - x_k\|^2 \mid g_i(x) + \frac{1}{2\alpha} \|x - x_k\|^2 \leq \tau \right\}$$

- α: Stepsize which depends on the weak convexity
- τ: Feasibility tolerance, $\|x_{k+1} - x_k\| \geq \sqrt{2\alpha\tau}$ implies $g_i(x_{k+1}) \leq 0$

Fritz-John (FJ) Stationarity

FJ Stationarity

A feasible x^* is an FJ point if $\exists \gamma_0^* \in \mathbb{R}_+, \gamma^* = (\gamma_1^*, ..., \gamma_m^*)^T \in \mathbb{R}_+^m$, $\zeta_f \in \partial f(x^*), \zeta_{gi} \in \partial g_i(x^*)$, $(\gamma_0^*, \gamma^*) \neq 0$

$$\gamma_0^* \zeta_f + \sum_{i=1}^m \gamma_i^* \zeta_{gi} \in -N_X(x^*), \quad \gamma_i^* g_i(x^*) = 0, \ \forall i = 1, ..., m.$$

Approximate FJ Stationarity

(a) A feasible x is an ϵ-FJ point if $\exists \gamma_0 \in \mathbb{R}_+, (\gamma_1, ..., \gamma_m)^T \in \mathbb{R}_+^m$, $\zeta_f \in \partial f(x), \zeta_{gi} \in \partial g_i(x), \sum_{i=0}^m \gamma_i = 1$

$$\mathrm{dist}(\gamma_0 \zeta_f + \sum_{i=1}^m \gamma_i \zeta_g, -N_X(x)) \leq \epsilon, \quad |\gamma_i g_i(x)| \leq \epsilon^2 \ \forall i = 1, ..., m.$$

(b) A feasible point x is an (ϵ, η)-FJ point if there exists an ϵ-FJ point x' such that $\|x - x'\| \leq \eta$.

Constraint Qualification (CQ)

Mangasarian-Fromovitz Constraint Qualification (MFCQ)

Let $A(x) = \{i \mid g_i(x) = 0, i = 1, ..., m\}$. MFCQ holds at x^* if

$$\exists v \in -N_X^*(x^*) \text{ s.t. } \zeta_{gi}^T v < 0 \ \forall i \in A(x^*), \forall \zeta_{gi} \in \partial g_i(x^*).$$

- MFCQ indicates the existence of a Slater point

σ-strong MFCQ

σ-strong MFCQ holds at x if $\exists \sigma > 0$

$$\exists v \in -N_X^*(x), \|v\| = 1 \text{ s.t. } \zeta_{gi}^T v \leq -\sigma \ \forall i \in A(x), \forall \zeta_{gi} \in \partial g_i(x).$$

Karush-Kuhn-Tucker (KKT) Stationarity

KKT Stationarity

A feasible x^* is a KKT point if $\exists \lambda^* = (\lambda_1^*, ..., \lambda_m^*)^T \in \mathbb{R}_+^m$, $\zeta_f \in \partial f(x^*), \zeta_{gi} \in \partial g_i(x^*)$

$$\zeta_f + \sum_{i=1}^m \lambda_i^* \zeta_{gi} \in -N_X(x^*), \quad \lambda_i^* g_i(x^*) = 0, \ \forall i = 1, ..., m.$$

Approximate KKT Stationarity

(a) A feasible x is an ϵ-KKT point if $\exists \lambda^* = (\lambda_1^*, ..., \lambda_m^*)^T \in \mathbb{R}_+^m$, $\zeta_f \in \partial f(x^*), \zeta_{gi} \in \partial g_i(x^*)$

$$\mathrm{dist}(\zeta_f + \sum_{i=1}^m \lambda_i \zeta_{gi}, -N_X(x)) \le \epsilon, \quad |\lambda_i g_i(x)| \le \epsilon^2 \ \forall i = 1, ..., m.$$

(b) A feasible point x is an (ϵ, η)-KKT point if there exists an ϵ-KKT point x' such that $\|x - x'\| \le \eta$.

Related Work and Outline

Related work:

- Inexact proximal point methods in [Ma, Lin, Yang 2020] and [Boob, Deng, Lan 2023]

Outline of our developments:

- Attain KKT of FJ stationarity with or without CQ
- Always feasible iterates
- Not requiring boundedness of domain

Double-loop Algorithm Structure

$$\begin{cases} \min_{x \in X} & f(x) \\ \text{s.t.} & g(x) := \max_{i=1,\dots,m} g_i(x) \le 0. \end{cases}$$

- $f(x)$ is ρ-weakly convex
- $g(x)$ is ρ-weakly convex if each $g_i(x)$ is ρ-weakly convex
- $f_{lb} = \inf_{x \in X} f(x) > -\infty$ and $g_{lb} = \inf_{x \in X} g(x) > -\infty$
- For any $x \in X$, $\zeta_f \in \partial f(x)$, $\zeta_g \in \partial g(x)$: $\|\zeta_f\|, \|\zeta_g\| \le M$

In each outer loop, we solve a $(\hat{\rho} - \rho)$-strongly convex functional constrained subproblem

$$\begin{cases} \min_{x \in X} & F_k(x) := f(x) + \frac{\hat{\rho}}{2}\|x - x_k\|^2 \\ \text{s.t.} & G_k(x) := g(x) + \frac{\hat{\rho}}{2}\|x - x_k\|^2 \le 0. \end{cases}$$

Inner Layer: Switching Subgradient Method (SSM)

$$\begin{cases} \min_{z\in Z} & F(z) \\ \text{s.t.} & G(z) \leq 0. \end{cases}$$

Definition

z is a (δ, τ)-optimal solution if $F(z) - F(z^*) \leq \delta$ and $G(z) \leq \tau$.

Algorithm The Switching Subgradient Method (SSM)

Input: $\tau > 0$, $T > 0$, $z_0 \in Z$, $\{\alpha_t\}_{t=0}^{T-1}$
Set $I = \emptyset$, $J = \emptyset$
for $t = 0, 1, ..., T-1$ **do**
If $G(z_t) \leq \tau$: $z_{t+1} = \text{proj}_Z(z_t - \alpha_t \zeta_{Ft})$, $\zeta_{Ft} \in \partial F(z_t)$, $I = I \cup \{t\}$; else: $z_{t+1} = \text{proj}_Z(z_t - \alpha_t \zeta_{Gt})$, $\zeta_{Gt} \in \partial G(z_t)$, $J = J \cup \{t\}$
end for
Output: $\bar{z}_T = \frac{\sum_{t\in I}(t+1)z_t}{\sum_{t\in I}(t+1)}$

NonLipschitz Conditions

- Previous convergence analyses on SSM [Lan and Zhou 2020, Ma, Lin, Yang 2020] assume uniform Lipschitz continuity for both $F(z)$ and $G(z)$
- Uniform Lipschitz continuity NOT hold for our subproblems!
    - Z can be unbounded
    - $F(z), G(z)$ are strongly convex

Consider the weaker nonLipschitz conditions [Grimmer 2019]: $\forall \tau > 0, \exists L_0, L_1 \geq 0$ such that $\forall z_1 \in \{z \mid G(z) \leq \tau\}, z_2 \in \{z \mid G(z) > \tau\}, \zeta_F \in \partial F(z_1),\ \zeta_G \in \partial G(z_2)$

$$\|\zeta_F\|^2 \leq L_0^2 + L_1(F(z_1) - F(z^*)),$$
$$\|\zeta_G\|^2 \leq L_0^2 + L_1(G(z_2) - G(z^*)).$$

- Subgradients bounded by suboptimality/infeasibility
- $F(z)$ and $G(z)$ are L_0-Lipschitz when $L_1 = 0$
- Allows $F(z)$ and $G(z)$ to grow quadratically when $L_1 > 0$

Convergence Results for SSM

Theorem

Given $\alpha_t = \frac{2}{\mu(t+2)+L_1^2/[\mu(t+1)]}$, $\tau > 0$, z_0 with $G(z_0) \le \tau$, we attain a (τ,τ)-optimal solution when $T \ge \max\left\{\frac{8L_0^2}{\mu\tau}, \sqrt{\frac{2L_1^2\|z_0 - z^*\|^2}{\mu\tau}}\right\}$.

Lemma

For any feasible $x_k \in X$, the non-Lipschitz condition holds for the subproblems with $L_0^2 = 9M^2 - 6\hat{\rho}g_{lb}$ and $L_1 = 6\hat{\rho}$.

Corollary

With $z_0 = x_k$, $\mu = \hat{\rho} - \rho$, $\alpha_t = \frac{2}{(\hat{\rho}-\rho)(t+2)+36\hat{\rho}^2/[(\hat{\rho}-\rho)(t+1)]}$, $\tau > 0$, we attain a (τ,τ)-optimal solution when

$$T \ge \max\left\{24(3M^2 - 2\hat{\rho}g_{lb})/(\mu\tau), \sqrt{72\hat{\rho}^2 D^2/(\mu\tau)}\right\}.$$

Outer Layer: Proximally Guided Switching Subgradient Method

Algorithm The Proximally Guided Switching Subgradient Method

Input: $\hat{\rho} > \max\{\rho, 1\}$, $\tau > 0$, T_{inner}, $x_0 \in X$ with $g(x_0) \leq 0$.
Set $\mu = \hat{\rho} - \rho$ and $\alpha_t = \frac{2}{(\hat{\rho}-\rho)(t+2)+\frac{36\hat{\rho}^2}{(\hat{\rho}-\rho)(t+1)}}$
for $k = 0, 1, ...,$ **do**
Set x_{k+1} as the output of $SSM(\tau, T_{inner}, x_k, \{\alpha_t\}_{t=0}^{T_{inner}})$ for the subproblem
end for

Boundedness of the Lagrange Multiplier

Assumption 1

The σ-strong MFCQ condition is satisfied for any subproblem.

Define $D := \sqrt{-8g_{lb}/(\hat{\rho} - \rho)}$ as an upper bound for the diameter of $\{x \mid G_k(x) \leq 0\}$

Lemma

Under Assumption 1, the optimal Lagrange multipliers for the subproblems are uniformly upper bounded by

$$B := \frac{M + \hat{\rho} D}{\sigma}.$$

Feasible Sequence of Iterates

$$\begin{cases} \tau_{FJ} = \frac{(\hat{\rho}-\rho)\epsilon^2}{4\hat{\rho}(2\hat{\rho}-\rho)} \\ T_{FJ} = \max\left\{\frac{96\hat{\rho}(2\hat{\rho}-\rho)(3M^2-2\hat{\rho}g_{lb})}{(\hat{\rho}-\rho)^2\epsilon^2}, \sqrt{\frac{288\hat{\rho}^3(2\hat{\rho}-\rho)D^2}{(\hat{\rho}-\rho)^2\epsilon^2}}\right\} \end{cases},$$

$$\begin{cases} \tau_{KKT} = \frac{(\hat{\rho}-\rho)\epsilon^2}{4(1+B)^2\hat{\rho}(2\hat{\rho}-\rho)} \\ T_{KKT} = \max\left\{\frac{96(1+B)^2\hat{\rho}(2\hat{\rho}-\rho)(3M^2-2\hat{\rho}g_{lb})}{(\hat{\rho}-\rho)^2\epsilon^2}, \sqrt{\frac{288(1+B)^2\hat{\rho}^3(2\hat{\rho}-\rho)D^2}{(\hat{\rho}-\rho)^2\epsilon^2}}\right\} \end{cases}.$$

Lemma

(a) For any $\hat{\rho} > \max\{\rho, 1\}$ with τ_{FJ} and T_{FJ}, we guarantee $g(x_k) \leq 0$ before x_k is an (ϵ, ϵ)-FJ point;

(b) Under Assumption 1, for any $\hat{\rho} > \max\{\rho, 1\}$ with τ_{KKT} and T_{KKT}, we guarantee $g(x_k) \leq 0$ before x_k is an (ϵ, ϵ)-KKT point.

Overall Convergence Results

Theorem: FJ Stationarity

For any $\hat{\rho} > \max\{\rho, 1\}$ with $\tau_{FJ} \in \mathcal{O}(\epsilon^2)$ and $T_{FJ} \in \mathcal{O}(1/\epsilon^2)$, we attain an (ϵ, ϵ)-FJ point for the original problem in $K \in \mathcal{O}(1/\epsilon^2)$ outer iterations.

Theorem: KKT Stationarity

Under Assumption 1, for any $\hat{\rho} > \max\{\rho, 1\}$ with $\tau_{KKT} \in \mathcal{O}(\epsilon^2)$ and $T_{KKT} \in \mathcal{O}(1/\epsilon^2)$, we attain an (ϵ, ϵ)-KKT point for the original problem in $K \in \mathcal{O}(1/\epsilon^2)$ outer iterations.

With or without Slater points, we attain the approximate FJ or KKT stationarity in $\mathcal{O}(1/\epsilon^4)$ subgradient evaluations.

Sparse Phase Retrieval under SCAD Constraints

Consider the experimental example:

$$\begin{cases} \min_{x\in X} & f(x) = \frac{1}{m}\sum_{i=1}^{m} |(a_i^T x)^2 - b_i^2| \\ \text{s.t.} & g(x) = \sum_{i=1}^{n} s(x_i) - p \le 0\ . \end{cases}$$

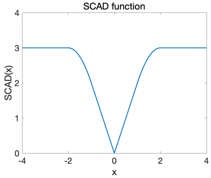
The SCAD function is defined as

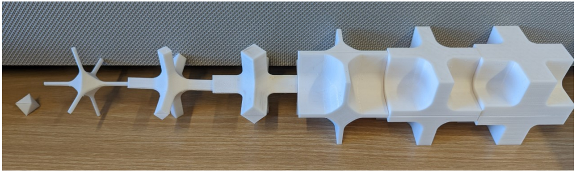
$$s(x_i) := \begin{cases} 2|x_i| & 0 \le |x_i| \le 1, \\ -x_i^2 + 4|x_i| - 1 & 1 < |x_i| \le 2, \\ 3 & |x_i| > 2. \end{cases}$$

- $X = [-10, 10]^n$, $m = 120$, $n = 120$
- $A \in \mathbb{R}^{m\times n}$, $x^* \in \mathbb{R}^n$, noise $\eta \in \mathbb{R}^m$ randomly sampled
- $b^2 = (Ax^*)^2 + \eta$
- $p \in [0, 3n)$ controls the sparsity

SCAD Function and Constraint Qualification (CQ)

1D SCAD function and seven 3D SCAD level sets with $p \in \{2.5, 3.5, 4.5, 5.5, 6.5, 7.5, 8.5\}$:

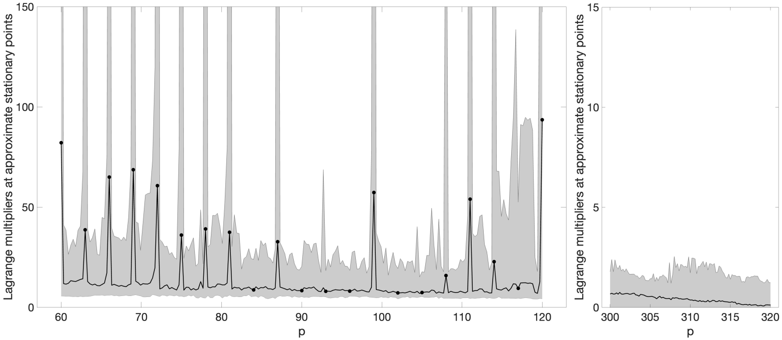




- A Slater point exists for all the subproblems if and only if p is a multiple of three!

SCAD Function and Constraint Qualification (CQ)

Small and large values of p:



- Over 30 independent replicates, approximately 5% has the Lagrange multipliers diverge when p is a multiple of three

Evaluate Stationarity and the Lagrange Multipliers

Evaluate the Stationarity:

- $\|\gamma_{k0}\zeta_{fk} + \gamma_k\zeta_{gk}\| = \hat{\rho}\|\hat{x}_{k+1} - x_k\| \approx \hat{\rho}\|x_{k+1} - x_k\|$
- $\|\zeta_{fk} + \lambda_k\zeta_{gk}\| = (1+\lambda_k)\hat{\rho}\|\hat{x}_{k+1} - x_k\| \approx (1+\lambda_k)\hat{\rho}\|x_{k+1} - x_k\|$

Evaluate the Lagrange multipliers:

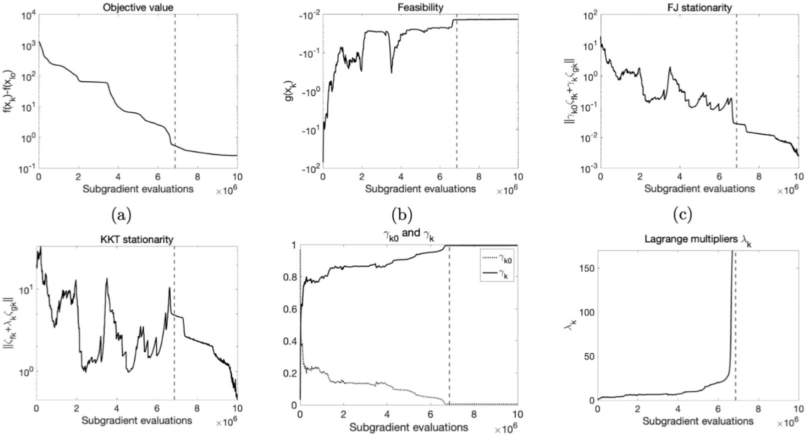
$$\frac{\sum_{t\in I_F}\alpha_t\zeta_{ft} + \sum_{t\notin I_F}\alpha_t\zeta_{gt}}{\sum_t \alpha_t} \approx \frac{\sum_{t\in I_F}\alpha_t\zeta_{Ft} + \sum_{t\notin I_F}\alpha_t\zeta_{Gt}}{\sum_t \alpha_t}$$

$$= \frac{x_k - x_{k+1}}{\sum_t \alpha_t} \approx \frac{x_k - \hat{x}_{k+1}}{\sum_t \alpha_t} \approx 0$$

- $\gamma_{k0} \approx \frac{\sum_{t\in I_F}\alpha_t}{\sum_t \alpha_t}, \quad \gamma_k \approx \frac{\sum_{t\notin I_F}\alpha_t}{\sum_t \alpha_t}, \quad \lambda_k \approx \frac{\sum_{t\notin I_F}\alpha_t}{\sum_{t\in I_F}\alpha_t}$

Experimental Results

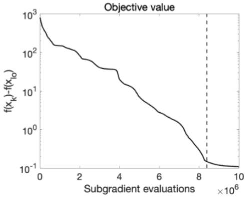
$K = 10^3, T = 10^4, p = 90$:



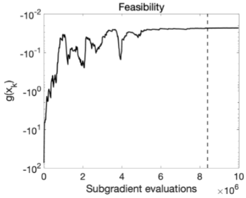
(a) (b) (c) (d) (e) (f)

Experimental Results

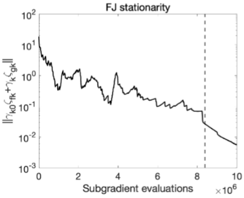
$K = 10^3, T = 10^4, p = 91$:



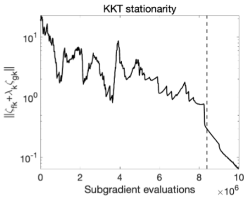
(a)



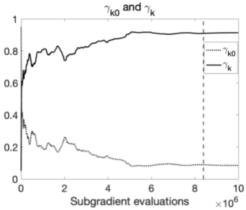
(b)



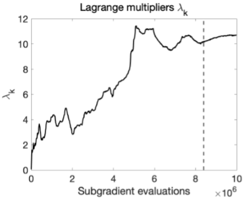
(c)



(d)



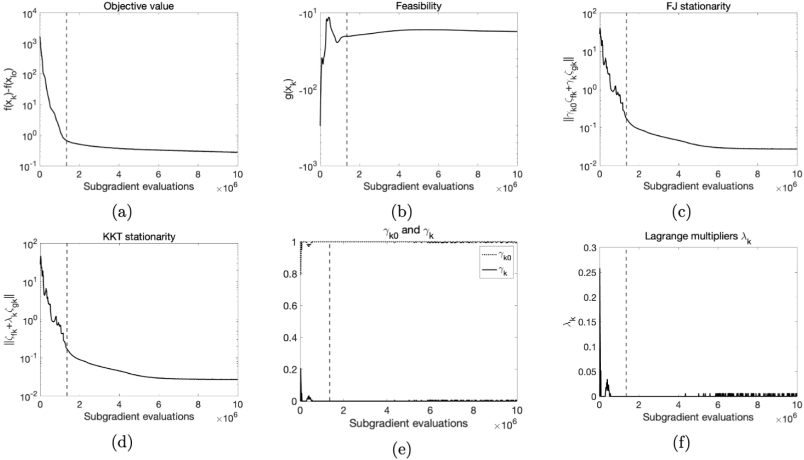
(e)



(f)

Experimental Results

$K = 10^3, T = 10^4, p = 320$:



Summary

- Convergence guarantee for KKT or FJ stationarity with or without Constraint Qualification
- Always feasible iterates
- Convergence guarantee without requiring domain compactness

Reference: Z. Jia and B. Grimmer, First-Order Methods for Nonsmooth Nonconvex Functional Constrained Optimization with or without Slater Points, arXiv Pre-print 2212.00927.